Enhanced Stability Regions for Model Predictive Control of Nonlinear Process Systems

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The problem of predictive control of nonlinear process systems subject to input constraints is considered. The key idea in the proposed approach is to use control-law independent characterization of the process dynamics subject to constraints via model predicative controllers to expand on the set of initial conditions for which closed–loop stability can be achieved. An application of this idea is presented to the case of linear process systems for which characterizations of the null controllable region (the set of initial conditions from where closed–loop stability can be achieved subject to input constraints) are available, but not practically implementable control laws that achieve stability from the entire null controllable region. A predictive controller is designed that achieves closed–loop stability for every initial condition in the null controllable region. For nonlinear process systems, while the characterization of the null controllable region remains an open problem, the set of initial conditions for which a (given) Lyapunov function can be made to decay is analytically computed. Constraints are formulated requiring the process to evolve within the region from where continued decay of the Lyapunov function value is achievable and incorporated in the predictive control design, thereby expanding on the set of initial conditions from where closed–loop stability can be achieved. The proposed method is illustrated using a chemical reactor example, and the robustness with respect to parametric uncertainty and disturbances demonstrated via application to a styrene polymerization process. © 2008 American Institute of Chemical Engineers *AIChE J,* 54: 1487–1498, 2008
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Introduction

The operation and control of chemical processes often encounters constraints that arise out of physical limitations on the control actuators. The constraints, if not accounted for in the control design, can cause performance deterioration or even instability in the closed–loop system. Specifically, the presence of constraints limits the set of initial conditions from where a process can be stabilized at a desired equilibrium point (the so-called null controllable region). A meaningful measure of how well the available control effort is being utilized by the control law can be obtained via a comparison of the stability region under a given control law with the null controllable region. Such a measure also provides assurance on the ability of the control law in recovering from the effect of disturbances that may temporarily drive the process away from the nominal operating point. These considerations have motivated extensive research on account-

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ing for constraints via modifications in existing control approaches (e.g., antiwindup designs[1]), as well as fostered the development of controllers that explicitly account for the presence of constraints via Lyapunov-based (see, for example,[2–7] and[8,9] for excellent reviews), and model-predictive control designs (see, for example,[10–17] and the survey paper, Ref. 18).

Given that process dynamics are sometimes identified or approximated by linear process systems, extensive research work has focused on designing and analyzing controllers that utilize a linear process description in computing the control action. Characterization of the null controllable region for linear process systems, while being difficult, is a tractable problem, and has been the focus of several research efforts.[19–22] Furthermore, several controller designs have been proposed that allow the possibility of turning any given *subset* of the null controllable region into the stability region of a proposed controller design.[23–25] Model predictive control (MPC) approaches allows implementation of stability constraints demanding the state to go to some invariant neighborhood of the origin (or the origin itself).[10,24] When guaranteeing feasibility from a subset of the null controllable region (under the assumption of initial feasibility of the optimization problem), the results use the approach of quantifying (and using as the horizon) the maximum over the minimum possible time for every point in the given set to be driven to the origin. Such an approach leads to prohibitively large values of the horizon (leading to practically un-implementable controllers) when requiring enhancement in the stability region. For some classes of linear systems (systems with real eigenvalues, low-order systems with complex eigenvalues), explicit expressions for the boundary of the null controllable region, parameterized by the magnitude of input constraints, have recently been characterized.[22] The work in Ref. 22 however, does not consider the problem of determining the control law that can stabilize all initial conditions in the null controllable region. As a special case of the key idea in the proposed work, we show how the characterization developed in[22] can be utilized within the model predictive control framework to achieve stabilization from all initial conditions in the null controllable region, without unduly increasing the computation complexity of the optimization problem.

For nonlinear processes, the problem of explicitly characterizing the null controllable region remains intractable. Lyapunov-based control designs address the problem of explicit characterizations of the stability region (see, e.g., Refs. 2,5,6) under given control laws. The stability regions, however, are limited to (possibly conservative estimates of) invariant subsets (Ω) of the set of states for which the Lyapunov function (V) can be made to decay (Π) under the specific control law. In the model predictive control framework, several designs have been proposed that guarantee closed–loop stability contingent on the assumption of initial feasibility of the optimization problem.[15,26–34] In[16,35,36] the optimization problem is reformulated within the framework of solving linear matrix inequalities (LMIs), with the ensuring "error" due to local linearization accounted for via robust MPC formulation, and stability is guaranteed under the assumption of existence of a solution to the LMIs. In,[37] Lyapunov-based and model predictive approaches were utilized within a switched controller framework to enable implementation of existing model predictive controllers with guaranteed stability region. More recently, in[38,39] (see Ref. 9 for further results and references), the stability properties of auxiliary Lyapunov-based controllers of[2,6] were utilized in formulating stability constraints in the optimization problem in a way that the predictive controllers of[38,39] mimic the (possibly conservative) stability region of the auxiliary control designs. The stability region estimates in the existing designs,[38,39] however, do not fully utilize the constraint handling properties of the predictive controller approach to expand on the set of initial conditions from where closed–loop stability can be achieved.

Motivated by these considerations, this work considers the problem of control of nonlinear process systems subject to input constraints and presents predictive controllers that utilize control law independent analysis of the process dynamics in the controller design. First, linear process systems are considered and a predictive controller is designed that achieves closed–loop stability for every initial condition in the null controllable region (not just subsets of the null controllable region) without resorting to (practically) infinite horizon. For nonlinear process systems, the set of initial conditions for which $\dot{V} < 0$ is achievable (subject to constraints, and independent of the control law) is first characterized. This characterization is then utilized to formulate constraints in the predictive controller that not only require the Lyapunov function value to decay, but also require the process to continue to evolve in the region from where successive decay in the Lyapunov function value is achievable. The enhancement in the set of initial conditions from where stability is achieved by the proposed method is illustrated using a chemical reactor example, and the robustness with respect to parametric uncertainty and disturbances demonstrated via a styrene polymerization process.

Preliminaries

In this section, we present the process description, a polymerization reactor to motivate our results and review existing Lyapunov-based predictive control designs.

Process description

We consider nonlinear processes with input constraints, described by:

$$\begin{aligned} \dot{x} &= f(x) + G(x)u(t) \\ u &\in \mathcal{U} \end{aligned} \tag{1}$$

where $x \in \mathrm{IR}^n$ denotes the vector of state variables, $u \in \mathrm{IR}^m$ denotes the manipulated inputs taking values in a nonempty convex subset $\mathcal{U}$ of IR^m, where $\mathcal{U} = \{u \in \mathrm{IR}^m: u_{\min} \leq u \leq u_{\max}\}$, $u_{\min} \in \mathrm{IR}^m$ and $u_{\max} \in \mathrm{IR}^m$ denote the lower and upper bounds on the manipulated input, $u^{\mathrm{norm}} > 0$ is such that $\|u\| \leq u^{\mathrm{norm}}$ implies $u \in \mathcal{U}$, where $\|\cdot\|$ is the Euclidean norm of a vector, and $f(0) = 0$. The vector function $f(x)$, and the matrix $G(x) = [g_1(x) \ldots g_m(x)]$ are assumed to be sufficiently smooth on their domains of definition. The notation L_fh denotes the standard Lie derivative of a scalar function $h(\cdot)$ with respect to the vector function $f(\cdot)$, ∂X denotes the boundary of a set X and $x(T^+)$ is used to denote the limit of the trajectory $x(t)$ as T is approached from the right, i.e.,

$x(T^+) = \lim_{t \to T^+} x(t)$. Throughout the manuscript, we assume that for any $u \in \mathcal{U}$ the solution of the system of Eq. 1 exists, and is continuous for all t. In the remaining of this section, we first present a styrene polymerization process that we will use to motivate our results, and then review a Lyapunov-based model predictive control design that mimics the stability region of Lyapunov-based nonlinear control designs.

Motivating example

To motivate our predictive control design methodology and to demonstrate an application of our results, we introduce in this section a polystyrene polymerization process. To this end, consider a model for polystyrene polymerization process given in[40] (also studied in, e.g.,[41])

$$
\begin{aligned}
\dot{C}_I &= \frac{(F_i C_{If} - F_t C_I)}{V_{pr}} - k_d C_I \\
\dot{C}_M &= \frac{(F_m C_{Mf} - F_t C_M)}{V_{pr}} - k_p C_M C_P \\
\dot{T} &= \frac{F_t (T_f - T)}{V_{pr}} + \frac{(-\Delta H)}{\rho c_p} k_p C_M C_P - \frac{hA}{\rho c_p V}(T - T_c) \\
\dot{T}_c &= \frac{F_c (T_{cf} - T_c)}{V_c} + \frac{hA}{\rho_c C_{pc} V_c}(T - T_c) \\
C_P &= \left[\frac{2 f k_d C_I}{k_t}\right]^{\frac{1}{2}} \\
k_d &= A_d e^{\frac{-E_d}{RT}} \\
k_p &= A_p e^{\frac{-E_p}{RT}} \\
k_t &= A_t e^{\frac{-E_t}{RT}}
\end{aligned}
\tag{2}
$$

where C_I, C_{If}, C_M, C_{Mf}, refer to the concentrations of the initiator and monomer in the inlet stream and in the reactor, respectively, T and T_f refer to the reactor and inlet stream temperatures, and T_c and T_{cf} refer to the coolant inlet and jacket temperatures, respectively. The primary manipulated inputs are the monomer and coolant flow rates. As is the practice with the operation of the polystyrene polymerization process,[40] the solvent flow rate is also changed in proportion to the monomer flow rate. The values of the process parameters are given in Table 1. The control objective is to stabilize the reactor at the unstable equilibrium point (C_i = 0.0480 kmolm^{-3}, C_M = 2.3331 kmolm^{-3}, T = 354.92 K, T_c = 316.2429 K), corresponding to the nominal values of the manipulated inputs of F_c = 0.000131 $m^3 s^{-1}$ and F_m = 0.000105 $m^3 s^{-1}$. The manipulated inputs are constrained as $0 \leq F_m \leq 0.003105$ $m^3 s^{-1}$, $0 \leq F_c \leq 0.0031$ $m^3 s^{-1}$. Owing to the high-dimensionality and nonlinearity of the process (as will be subsequently seen), stability region estimates derived using Lyapunov-based tools are conservative, and development of control designs that best use the available control action to enhance the set of initial conditions from where stabilization is achieved is of significance. We will demonstrate the application, as well as investigate the robustness, of the proposed control design to the styrene polymerization example, while illustrating the finer details of the proposed controller using an illustrative chemical reactor.

Table 1. Styrene Polymerization Parameter Values and Units

$F_i = 3 \times 10^{-4}$	$m^3 s^{-1}$
$F_m = 10.5 \times 10^{-4}$	$m^3 s^{-1}$
$F_s = 12.75 \times 10^{-4}$	$m^3 s^{-1}$
$F_t = 26.25 \times 10^{-4}$	$m^3 s^{-1}$
$F_c = 13.1 \times 10^{-4}$	$m^3 s^{-1}$
$C_{If,n} = 0.5888$	$kmolm^{-3}$
$C_I = 0.0480$	$kmolm^{-3}$
$C_{Mf,n} = 9.975$	$kmolm^{-3}$
$C_M = 2.3331$	$kmolm^{-3}$
$T_{f,n} = 306.71$	K
$T = 354.9205$	K
$T_{cf,n} = 294.85$	K
$T_c = 316.2429$	K
$A_d = 5.95 \times 10^{14}$	s^{-1}
$A_t = 1.25 \times 10^{10}$	s^{-1}
$A_p = 1.06 \times 10^{8}$	$kmolm^{-3} s^{-1}$
$E_d/R = 14.897 \times 10^{3}$	K
$E_t/R = 8.43 \times 10^{2}$	K
$E_p/R = 3.557 \times 10^{3}$	K
$f = 0.6$	
$\Delta H = -1.67 \times 10^{4}$	$kJkmol^{-1}$
$\rho c_p = 360$	$kJm^{-3}K^{-1}$
$hA = 700$	$JK^{-1}s^{1}$
$\rho_c c_{pc} = 966.3$	$kJm^{-3}K^{-1}$
$V_{pr} = 3.0$	m^3
$V_c = 3.312$	m^3

Lyapunov-based model predictive control

In this section, we briefly review a recent result on a Lyapunov-based predictive controller that has an explicitly characterized feasibility and stability region. To this end, consider the system of Eq. 1, for which a predictive controller[38] is designed of the form

$$u_{LBPC} = argmin\{J(x,t,u(\cdot))|u(\cdot) \in S\} \tag{3}$$

$$s.t. \quad \dot{x} = f(x) + G(x)u \tag{4}$$

$$\dot{V}(x(\tau)) \leq -\epsilon^* \quad \forall\, \tau \in [t, t+\Delta) \ if \ \ V(x(t)) > \delta' \tag{5}$$

$$V(x(\tau)) \leq \delta' \ \forall\, \tau \in [t, t+\Delta) \ \ if \ \ V(x(t)) \leq \delta' \tag{6}$$

where $S = S(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t + T]$ into U and T is the horizon. Eq. 4 is the nonlinear model describing the time evolution of the state x, V is a control Lyapunov function, and δ', ε^* are parameters to be determined. A control $u(\cdot)$ in S is characterized by the sequence $\{u[j]\}$ where $u[j] := u(j\Delta)$ and satisfies $u(\tau) = u[j]$ for all $\tau \in [t + j\Delta, t + (j + 1)\Delta)$. The performance index is given by

$$J(x,t,u(\cdot)) = \int_t^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u(s)\|_R^2\right] ds \tag{7}$$

where Q is a positive semidefinite symmetric matrix, and R is a strictly positive definite symmetric matrix. $x^u(s;x,t)$

denotes the solution of Eq. 1, due to control u, with initial state x at time t. The minimizing control $u^0(\cdot) \in S$ is then applied to the plant over the interval $[t, t + \Delta)$, and the procedure is repeated indefinitely.

The stability properties of the predictive controller are characterized using a bounded controller of the form (e.g., see Refs. 2,5 and 6)

$$u(x) = -k(x)(L_G V)'(x) \tag{8}$$

$$k(x) = \frac{L_f V(x) + \sqrt{\left(L_f V(x)\right)^2 + \left(u_{\max}\|(L_G V)'(x)\|\right)^4}}{\|(L_G V)'(x)\|^2\left[1 + \sqrt{1 + \left(u_{\max}\|(L_G V)'(x)\|\right)^2}\right]} \tag{9}$$

when $L_G V(x) \neq 0$ and $k(x) = 0$ when $L_G V(x) = 0$ where $L_f V(x) = \frac{\partial V(x)}{\partial x} f(x)$, $L_G V(x) = [L_{g_1} V(x) \cdots L_{g_m} V(x)]'$ and $g_i(x)$ is the i-th column of the matrix $G(x)$. For the controller of Eqs. 8 and 9, one can show, using a standard Lyapunov argument, that whenever the closed–loop state, x, evolves within the region described by the set

$$\Pi = \{x \in \mathrm{IR}^n : L_f V(x) \leq u^{\mathrm{norm}}\|(L_G V)'(x)\|\} \tag{10}$$

then the control law satisfies the input constraints, and the time-derivative of the Lyapunov function is negative-definite. An estimate of the stability region can be constructed using a level set of V, i.e.

$$\Omega = \{x \in \mathrm{IR}^n : V(x) \leq c^{\max}\} \tag{11}$$

where $c^{\max} > 0$ is the largest number for which $\Omega \subseteq \Pi$. Closed–loop stability and feasibility properties under the Lyapunov–based predictive controller are inherited from the bounded controller under discrete implementation, and are formalized in Theorem 1 following (for a proof, see Ref. 38).

Theorem 1[38]: *Consider the constrained system of Eq.* 1 *under the MPC law of Eqs.* 3–7. *Then, given any* $d \geq 0$, $x_0 \in \Omega$, *where* Ω *was defined in Eq.* 11, *there exist positive real numbers* δ', ε^*, Δ^*, *such that if* $\Delta \in (0, \Delta^*]$, *then the optimization problem of Eq.* 3–7 *is feasible for all times,* $x(t) \in \Omega$ *for all* $t \geq 0$ *and* $\limsup_{t\to\infty} \|x(t)\| \leq d$.

Remark 1: The key idea in the predictive control design is to identify stability constraints that can (a) be shown to be feasible, and (b) upon being feasible can guarantee stability. The analysis of discrete implementation of the control law of Eqs. 8 and 9 ensures the existence of a feasible solution to the predictive controller formulation from an explicitly characterized set of initial conditions. While the predictive controller utilizes the auxiliary control design to address the problem of guaranteeing initial feasibility, utilization of the constraint of Eq. 5 only imitates the stability properties (and the stability region) of the bounded controller, and does not fully exploit the constraint handling capabilities of the predictive control approach to expand on the set of initial conditions from where closed–loop stability is achieved.

Enhancing the Stability Region Estimates Using Model Predictive Control

The stability region estimates of existing Lyapunov-based predictive controllers are limited (and dependent upon) stability region estimates obtained using the auxiliary control approaches, and by not fully utilizing the constraint handling capabilities of the predictive control approach, suffer from the same possible conservatism as the auxiliary control designs. In this section, we present a predictive control design wherein constraints are formulated that, by better utilizing Lyapunov-based analysis tools, enhance the set of initial conditions from where closed–loop stability is achieved. Before we proceed to the controller design for nonlinear systems, we first consider, as a special case, linear systems subject to constraints, and show how the utilization of the process dynamics in the controller design results in a predictive controller that guarantees stabilization from all initial conditions for which closed–loop stability can be achieved subject to constraints. We next consider nonlinear systems and formulate a predictive controller that not only provides an explicit characterization of the stability region, but also enhances the set of initial conditions from which closed–loop stability is achieved.

Linear systems subject to constraints

Linear descriptions of the process dynamics are often utilized in controller design for chemical processes. While extensive results exist on constructing control designs that guarantee stability from any given subset of the null controllable region (see, e.g., Refs. 10,19–25,42), the computational complexity of the control design typically renders the control implementation impractical as larger and larger stability regions are desired. Furthermore, there exists a lack of results that guarantee stability for any initial condition in the entire null controllable region. In this section, we show how the characterization of the null controllable region, developed in,[22] can be utilized within the predictive control approach in achieving stability for all initial conditions in the null controllable region. To this end, consider processes whose dynamics can be described by

$$\dot{x}(t) = Ax(t) + Bu(t), \; u \in \mathcal{U} \tag{12}$$

where A and B are constant $n \times n$ and $n \times m$ matrices, respectively. A summary of characterization of the null controllable region is described following.[22]

Null controllable region for linear systems. A state x_0 is said to be null controllable if there exists a $T \in [0, \infty)$, and an admissible control $u(t)$, such that the state trajectory $x(t)$ of the system of Eq. 12 satisfies $x(0) = x_0$ and $x(T) = 0$, and the union of all null controllable sets is called the null controllable region of the system, which we denote by $X^{\max}$. The null controllable region characterized as (see Ref. 22) $X^{\max} = \bigcup_{T\in[0,\infty)}\{x = -\int_0^T e^{-A\tau}Bu(\tau)d\tau : u(\tau) \in \mathcal{U}\}$ can be shown to be a bounded convex open set containing the origin if A is unstable. It can be shown that the null controllable region of the multi-input system of Eq. 12 is the Minkowski sum of the single input subsystems

$$\dot{x}(t) = Ax(t) + b_i u_i(t), \; u_i(t) \in \mathcal{U}_i \tag{13}$$

where $B = [b_1\ b_2\ \ldots\ b_m]$ and u_i denotes the i-th component of the vector u. Specifically, let $X_i^{\max}$ denote the null controllable region of the subsystem of Eq. 13 then $X^{\max} = \sum_{i=1}^{m} X_i^{\max} = \{x_1 + x_2 + \cdots + x_m : x_i \in X_i^{\max}, i = 1, \ldots, m\}$. For systems with real eigenvalues (see Ref. 22 for computing the null controllable region for low-dimensional systems with complex eigenvalues), the boundary of the null controllable region can be computed as[22]

$$\partial X_i^{\max} = \left\{ \pm \left[\sum_{j=1}^{n-1} 2(-1)^j e^{-A(t-t_j)} + -1^n I \right] A^{-1} b_i u_i^{\text{norm}} : 0 \right. \\ \left. = t_1 \leq t_2 \leq \cdots \leq t_{n-1} \leq t \leq \infty \right\} \quad (14)$$

Equation 14 can be used to verify whether a state lies within the null controllable region and, more importantly, can be used to compute, for a given state, the unique value of u_i^*, such that the state resides on the boundary of the null controllable region of a system of the form of Eq. 13 with a constraint of u_i^* on the manipulated input u_i. Utilizing these properties, for a given state x_0 we define a function $\bar{u}_i^*(x_0)$ as the unique positive number u_i^* for which $x_0 \in \partial X_i^{\max}(u_i^*)$. Essentially, for a given state x_0, Eq. 14 is solved to yield t_i, $i = 2 \ldots n - 1$, t and u_i^{norm}. The value of u_i^{norm} equals the fictitious constraint u_i^* (see Eq. 22 for an illustrative example). In the next subsection, we show how the predictive control approach can utilize such a characterization in enabling stabilization from all points within the null controllable region.

Predictive control design with the null controllable region as the stability region. The key idea in the predictive control design is as follows: for any given value of the state, the value u_i^* represents the minimum control action required to stabilize the system. A meaningful control action, therefore, would be one that drives the process in a way that the minimum control action required to stabilize the system decreases. This intuitive idea is formulated mathematically in Theorem 2 later. To this end, consider the system of Eq. 12, and an $x_0 \in X^{\max}$. Let $x_{i,0} \in X_i^{\max}(u_i^*)$, $i = 1, \ldots, m$ be such that $x_0 = \sum_{i=1}^{m} x_{i,0}$, with $u_i^* \leq u_i^{\text{norm}}$. The predictive controller that guarantees stabilization from all initial conditions in $X^{\max}$ takes the form

$$u_{i,MPC} = \text{argmin}\{J(x, t, u(\cdot)) | u(\cdot) \in U,\ x(0) = x_{i,0}\} \quad (15)$$

$$s.t.\ \dot{x} = Ax + b_i u_i \quad (16)$$

$$\dot{\bar{u}}_i^*(x(t)) \leq 0 \quad (17)$$

Equation 16 is the linear model describing the time evolution of the state x, due to the i-th manipulated input. The performance index is given by

$$J(x, t, u(\cdot)) = \dot{\bar{u}}^*(x_i(t)) \quad (18)$$

The minimizing controls $u_i^0(\cdot)$ are then applied to the plant and the procedure is repeated indefinitely. Note that the aforementioned formulation is a continuous time version of the MPC, and assumes instantaneous evaluation and implementation of the computed control value. The result under continuous implementation is presented in Theorem 2, and the "implement and hold" approach demonstrated and discussed in the simulation example for linear systems, and addressed explicitly in the predictive control design for nonlinear process systems in Theorem 3.

Theorem 2: *Consider the constrained system of Eq.* 12 *under the MPC law of Eqs.* 15–18. *Then, given any* $x_0 \in X^{\max}$, *the optimization problem of Eq.* 15–18 *is feasible for all times, and* $\lim_{t\to\infty} x(t) = 0$.

Proof of Theorem 2: We first prove the results for a single input system, and then illustrate the generalization to multi-input systems. In the proof, the key things to show are guaranteed feasibility of the optimization problem and the optimal solution leading to closed–loop stability.

Single input system: In this part of the proof, we will drop the subscript on the input, with the understanding that a single input system is being analyzed. Consider an $x_0 \in X^{\max}$, for which $\bar{u}^*(x_0) = u_0^* < u^{\text{norm}}$. In part 1, we show feasibility of the optimization problem, and in part 2, the implementation of the optimal solution resulting in closed–loop stability.

Part 1: Since $x_0 \in X^{\max}(u^{\text{norm}})$, there exists at least one input trajectory $u(t)$ with $|u(t)| \leq u^{\text{norm}}$ such that $\lim_{t\to\infty} x(t) = 0$. Out of all such possible trajectories (for which $\lim_{t\to\infty} x(t) = 0$) let

$$u_1^* = \min_{|u_1(t)| \leq u^{\text{norm}}} \max_{t,\, x(0)=x_0} \bar{u}^*(x_{u_1}(t)) \quad (19)$$

where $x_{u_1}(t)$ denotes the state profile corresponding to an input profile of $u_1(t)$. Thus, u_1^* represents the minimum (over all possible stabilizing trajectories) of the maximum (over time) value that the function $\bar{u}^*(\cdot)$ takes. Note that if $u_1^* \geq u^{\text{norm}}$, then an x_1^*, such that $\bar{u}^*(x_1^*) = u_1^*$ will be such that $x_1^* \in X^{\max}(u^{\text{norm}})$ (in other words, it would mean that the process starting from a state outside the null controllable region is actually stabilized), which leads to a contradiction, we, therefore, have that

$$u_1^* < u^{\text{norm}} \quad (20)$$

Let $u_1^* = u_0^* + \gamma$ with $\gamma > 0$. Since $x_0 \in \partial X^{\max}(u_0^*)$, this implies that $x_0 \in X^{\max}(u_0^* + \gamma/2)$. Denoting

$$u_2^* = \min_{|u_2(t)| \leq u_0^* + \gamma/2} \max_{t,\, x(0)=x_0} \bar{u}^*(x_{u_2}(t)) \quad (21)$$

and invoking Eq. 20 again with $u_0^* + \gamma/2 = u^{\text{norm}}$, we get that $u_2^* < u_0^* + \gamma/2$. Furthermore, noting that the minimizations of Eqs. 19 and 21 are exactly the same, albeit with a larger constraint in Eq. 19 compared to Eq. 21, we get that $u_1^* = u_0^* + \gamma \leq u_2^* < u_0^* + \gamma/2$, which once again leads to a contradiction, implying γ cannot be a positive real number. This finally leads to the conclusion that for any $x_0 \in X^{\max}(u^{\text{norm}})$, there exists a manipulated input profile and corresponding state trajectory, such that $\bar{u}^*(x(t+\delta t)) \leq \bar{u}^*(x(t))$ for all $\delta t > 0$. This implies that along such a trajectory the function $\bar{u}^*(x(\cdot))$ is nonincreasing, implying the feasibility of the constraint $\dot{\bar{u}}^*(x(t)) \leq 0$.

Part 2: Having established the feasibility of the optimization problem in Part 1 previously, consider now an x_0 in $X^{\max}$ for which $J^*(x_0, t, u(\cdot)) = \min \dot{\bar{u}}^*(x_0(t)) = 0$. This implies that for this x_0, the minimizing u_{MPC} is such that the vector $Ax_0 + bu$ (which represents the current direction of the state trajectory) is on the tangent plane to the surface defining $\partial X^{\max}(\bar{u}^*(x_0))$. This would further imply that the vectors Ax_0 and bu_{MPC} must themselves be coplanar (if they were not, a different allowable value for u_{MPC} could have been chosen to point the vector $Ax_0 + bu$ away from the tangent plane to the surface defining $\partial X^{\max}(\overline{u}^*(x_0))$, resulting in a $\dot{J}^*(x_0(t)) < 0$). Upon implementation of such a u_{MPC}, the tangent to $\partial X^{\max}$ at $x(t^+)$ cannot remain in the same plane (due to the strict convexity of the boundary of the set $X^{\max}$) as that of the vector b resulting in $\min \dot{\bar{u}}^*(x_0^+) < 0$. Therefore, for any x_0 for which the minimum of $\dot{\bar{u}}^*(x(0)) = 0$, the minimum of $\dot{\bar{u}}^*(x(0^+)) < 0$ ensuring convergence of $\overline{u}^*(x(t))$ to zero, in turn resulting in $\lim_{t\to\infty} x(t) = 0$.

Multiple input system: The result for the multiple input system is a direct generalization for the single input system. Having defined $x_0 = \sum_{i=1}^{m} x_{i,0}$, $X^{\max}$ and $X_i^{\max}$, the evolution of the multiple input system is exactly the same as the sum of the multiple single input systems. Feasibility and stability of the subsystems yields stability for the original multi-input system.

Remark 2: While extensive results exist on stabilization of linear systems, the stability guarantees are provided for subsets (which can get arbitrarily close to the null controllable region) of the null controllable region, and the control design becomes practically impossible to implement as larger stability regions are sought. The approach in the existing results is to estimate the time that it would take for all initial conditions in the "desired" stability region to reach the origin, and to incorporate it in some fashion in the controller design. In model predictive control approaches, this idea can directly be utilized via large or variable horizon (e.g., see Refs. 10 and 24), leading to computationally expensive optimization problems. In all of these approaches, the idea remains the same: require the state to go to the origin (or some neighborhood of the origin) by some time (the horizon), and pick a large enough horizon to ensure feasibility of the optimization problem. When the horizon is variable, the optimization problem is in general difficult to solve, since the number of decision variables in the optimization problem itself keep changing. When the horizon is fixed, the number of decision variables that have to be retained grows as larger and larger subsets of the null controllable region are desired as the stability region. In contrast, the proposed predictive controller achieves guaranteed feasibility and stability for all initial conditions in the null controllable region. Note also that while the specific objective function of Eq. 18 is designed to satisfy the overriding requirement of stabilization (especially for initial conditions close to the boundary of the null-controllable region for which existing predictive control designs would result in a computationally unimplementable controller), once the state trajectory reaches closer to the desired equilibrium point (and inside the stability region of existing predictive controllers[38]), switching can be executed to implement the predictive controllers that allow for the minimization of a more general objective function.

Remark 3: Note that while the results of Theorem 2 are derived under the assumption of continuous implementation of the control action, in practice the results can be implemented when the control action is computed and held for a certain period of time (as in most applications). In doing so, for any given value of the state the current value of u_i^* is computed, and instead of computing a control action that yields $\dot{u}_i^* < 0$, a control action is computed that results in a lower value of u_i^* at the next sampling instant (see the simulation example for a demonstration), thereby, not requiring the computation and satisfaction of the constraint on the derivative of u_i^*.

Remark 4: The result achieving stabilization from the null controllable region can best be understood in light of the result using the control Lyapunov function. Specifically, the controller of Eqs. 3–18 does not guarantee stabilization from all initial conditions in the null controllable region due to the following reasons: (1) for a choice of a CLF V, $\dot{V}$ is not necessarily guaranteed to be negative for all initial conditions in $X^{\max}$, (2) even if a certain choice of the CLF resulted in $\dot{V}$ being negative for all initial conditions in $X^{\max}$, the level sets of a CLF may not necessarily coincide with the boundary of the null controllable region. The stability region estimate would, therefore, typically be a subset of the null controllable region. Note also that if it were always possible to obtain the global optimum to the optimization problem, then for the specific choice of the objective function, the constraint of Eq. 17 would be redundant. Specifically, if a control action were to exist that would make $\dot{\bar{u}}^*(x) \leq 0$ it would naturally be chosen over another control action for which $\dot{\bar{u}}^*(x) > 0$ (due to the specified objective function). In implementing the control algorithm, however, the optimization problem may not always be able to compute the global optimum. The constraint of Eq. 17 ensures that a local minima of $\dot{\bar{u}}^*(x)$, for which $\dot{\bar{u}}^*(x)$ may be greater than zero (and may lead to destabilization), is avoided, and only a stabilizing solution is chosen.

Simulation example. Consider a linear system of the form of Eq. 12 with $A = \begin{bmatrix} 0 & -0.5 \\ 1.0 & 1.5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ -1.0 \end{bmatrix}$ and $u^{\text{norm}} = 1$ (representing a nominally unstable linear system). The null controllable region $X^{\max}$, for this system is computed using Eq. 14 and is shown in Figure 1. For the sake of comparison, the stability region for the predictive controller of Theorem 1, computed using a quadratic Lyapunov function, the set Π (as described in Eq. 10), and then constructing the largest level set of the Lyapunov function $V(x) = c^{\max} = 1.1$ completely contained in Π is also shown, denoted by Ω. The conservativeness in using level sets of the Lyapunov function to estimate the stability region (for this particular example) is seen via the part of $X^{\max}$ not captured in Ω. While the theoretical results are derived under the assumption of continuous implementation of the control action, the simulation results demonstrate the discrete implementation of the controller, with a discretization time of $\Delta = 0.1$. The constraint of Eq. 17, is, therefore, implemented as $\overline{u}^*(x(t+\Delta)) \leq \overline{u}^*(x(t))$. The function $\overline{u}^*(x)$ is evaluated by computing the unique solution pair u^*, T to the equation (utilizing Eq. 19 in Ref. 22)

$$x(t) = (2e^{AT} + I)A^{-1}Bu^* \quad (22)$$

and then by evaluating $\overline{u}^*(\cdot) = |u^*|$. Note that the same equation, setting $u^* = 1$, and by varying T from 0 to ∞, is used to construct the boundary of the set $X^{\max}$ (for more details, see Ref. 22). The optimization problem in the predictive controller formulation is solved by using the MATLAB function FMINCON.

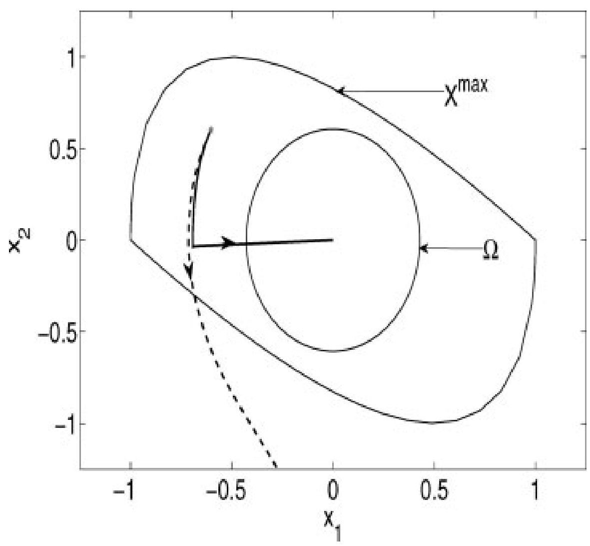

Figure 1. Evolution of the state trajectory for the linear system example under the predictive controller of Eqs. 3–7 (dashed line) with a stability region Ω, and under the proposed predictive controller (solid line) with a stability region X^{max}.

To illustrate the stabilization properties of the proposed predictive controller, we pick an initial condition x_0 = [−0.6032, 0.6003] in X^{max}, and try to stabilize it using the predictive controller of Eqs. 3–5 (that requires the control action to result in a decay in the value of $V(x)$). As can be seen from the dashed lines in Figure 1, closed–loop stability is not achieved (the corresponding state trajectories and input profile can be seen as dashed lines in Figure 2 a–c). In contrast, when the control action computed by the proposed predictive controller is implemented, closed–loop stability is achieved. Note that the proposed predictive controller does not try to compute a control action that decreases the value of the Lyapunov function, but instead computes a control action that drives the state trajectory along lower "level sets" of u^*. Figure 2 d–e shows the evolution of the Lyapunov function, and that of $\overline{u}^*(x(t))$ for the two scenarios. Once again, the figures demonstrate the decrease in the value of the Lyapunov function initially achievable (see inset), after which the state escapes the set of initial conditions from where the negative definiteness of $\dot{V}$ can be enforced. In contrast, the solid lines show the decrease in the value of $\overline{u}^*(x(t))$ enforced by the predictive controller (note that the predictive controller also enforces a continual decrease in the value of the Lyapunov function is only incidental). In summary, the proposed predictive controller drives the state trajectory to successively lower values of $\overline{u}^*(x(t))$ eventually stabilizing the system.

Model predictive control of nonlinear systems

In contrast to linear systems, where an explicit characterization of the null controllable region is possible, for nonlinear process systems such a characterization remains an open problem. In[39], predictive controllers were designed that utilized auxiliary Lyapunov–based control design for estimating

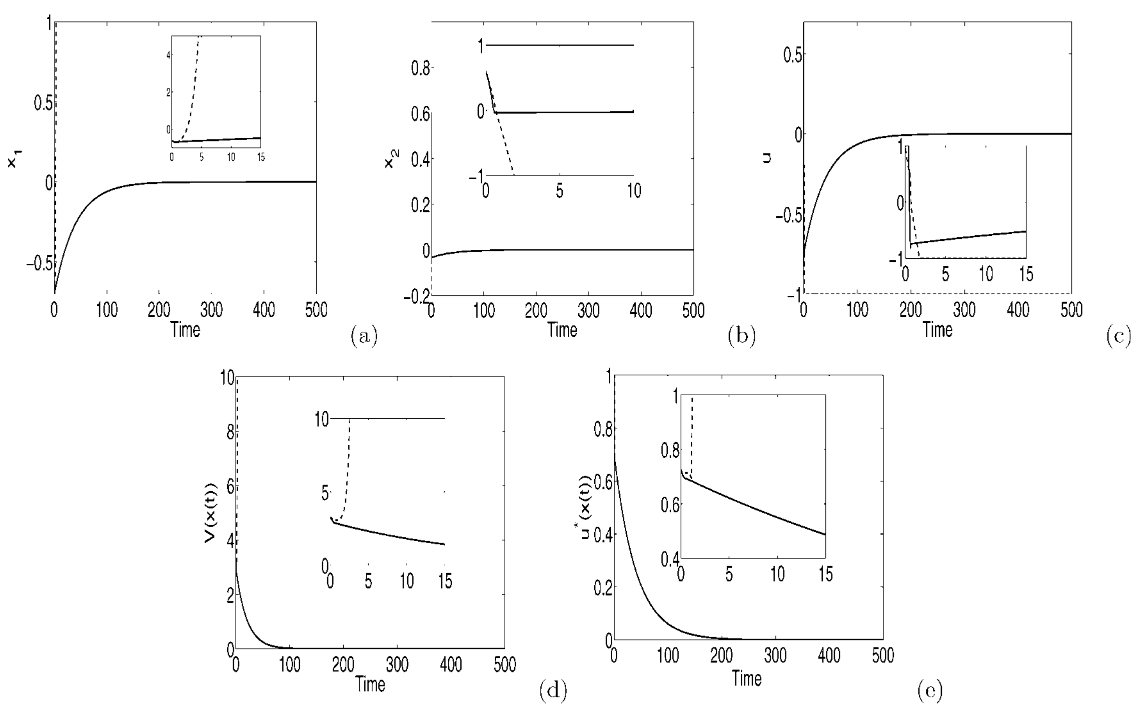

Figure 2. The state (a-b), and input profiles (c), and the evolution of the Lyapunov function (d), and $u^*(x(t))$ (e), for the linear system example under the predictive controller of Eqs. 3–7 (dashed lines), and under the proposed predictive controller (solid lines).

The insets show the initial evolution of the system.

the feasibility and stability region. In the predictive control design of[39], the first layer of conservativeness stems from the estimation of Π, which only captures initial conditions for which negative definiteness of $\dot{V}$ can be achieved by the auxiliary control law, instead of characterizing the set of initial conditions for which negative definiteness of $\dot{V}$ can be achieved independent of the control law (which we will characterize and denote by Π^+). Additionally, only requiring $\dot{V}$ to be negative allows stabilization from all initial conditions inside Ω, but misses out on achieving stabilization from initial conditions outside Ω, but inside Π.

Nonlinear model predictive controller. We utilize in this section the constraint handling capabilities of the predictive controller to expand on the set of initial conditions from where closed–loop stability can be achieved to alleviate the possible conservatism associated with Lyapunov-based control designs. To this end, we first characterize the set Π^+, for which negative definiteness of the Lyapunov function derivative can be achieved subject to manipulated input constraints (and independent of the control law) described by

$$\Pi^+ = \{x \in \mathrm{IR}^n : L_f V(x) - \sum_{i=1}^{m} |L_{g_i} V(x)| u_i^{\mathrm{norm}} \leq -\epsilon^{**}\} \quad (23)$$

where ε^{**} is a positive number to be defined. The set Π^+, therefore, denotes the entire set of initial conditions from where $\dot{V} < -\varepsilon^{**}$ is achievable (and not just the set from where a specific control law can achieve $\dot{V} < 0$, thereby improving upon the estimate Π in Eq. 10). The idea behind the expression in Eq. 23 is as follows: each element of the vector $L_G V(x)$, denoted by $L_{g_i} V(x)$ captures the effect of the i-th component of the manipulated input on the Lyapunov function derivative. The term $-|L_{g_i} V(x)| u_i^{\mathrm{norm}}$, therefore, captures the most that the i-th manipulated input can contribute toward making $\dot{V}(x)$ negative. Alternatively, the expression can also be thought of as the set of states for which $\dot{V}(x)$ is negative under the "bang-bang" control law given by $u_i(x) = -sgn(L_{g_i} V(x)) u_i^{\mathrm{norm}}$ where $sgn(x) = 1$ if $x \geq 0$ and $sgn(x) = -1$ if $x < 0$. By accounting for the maximum control action available, the set Π^+ expands on the estimate Π. Subsequently, computation of the largest level set Ω^+, of the form

$$\Omega^+ = \{x \in \mathrm{IR}^n : V(x) \leq c^{\max^+}\} \quad (24)$$

completely contained in Π^+ improves upon the estimate Ω. Requiring $\dot{V} \leq -\varepsilon^{**}$ instead of only requiring $\dot{V} < 0$ is formulated to ensure stabilization subject to implement and hold (similar to the result in Theorem 1). Having defined the sets Π^+ and Ω^+ the predictive controller enhancing the set of initial conditions from which stability is achieved (accounting specifically for initial conditions outside Ω^+ but inside Π^+) takes the form

$$u = \mathrm{argmin}\{J(x, t, u(\cdot)) | u(\cdot) \in S\} \quad (25)$$

$$s.t. \;\; \dot{x} = f(x) + G(x)u \quad (26)$$

$$\dot{V}(x(\tau)) \leq -\epsilon^* \;\; \forall\, \tau \in [t, t+\Delta) \; if \;\; V(x(t)) > \delta' \quad (27)$$

$$V(x(\tau)) \leq \delta' \;\forall\, \tau \in [t, t+\Delta) \;\; if \;\; V(x(t)) \leq \delta' \quad (28)$$

$$x(t+\tau) \in \Pi^+ \;\forall\, \tau \in [t, t+\Delta) \; if \;\; V(x(t)) > c^{\max^+} \quad (29)$$

where $S = S(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t,t+T]$ into U and T is the horizon. Eq. 26 is the model describing the time evolution of the state x under continuous operation, V is the control Lyapunov function (CLF) and δ', $\varepsilon^* > 0$ are parameters defined in Theorem 1. A control $u(\cdot)$ in S is characterized by the sequence $\{u[j]\}$ where $u[j] := u(j\Delta)$ and satisfies $u(\tau) = u[j]$ for all $\tau \in [t+j\Delta, t+(j+1)\Delta)$.

The performance index is given by

$$J(x, t, u(\cdot)) = \int_{t}^{t+T} \left[\|x^u(s; x, t)\|_Q^2 + \|u(s)\|_R^2\right] ds + vV(x(t+\Delta)) \quad (30)$$

where Q is a positive semidefinite symmetric matrix, R is a strictly positive definite symmetric matrix, and $v > 0$. $x^u(s;x,t)$ denotes the solution of Eq. 1, due to control u, with initial state x at time t. The minimizing control $u^0(\cdot) \in S$ is then applied to the process over the interval $[t, t+\Delta)$ and the procedure is repeated indefinitely. The feasibility and stability properties of the predictive controller are formalized in Theorem 3 following:

Theorem 3: *Consider the constrained system of Eq.* 1 *under the MPC law of Eqs.* 25–30. *Then, given any* $d > 0$, *there exists a positive real number* ε^{**} *such that if* $x_0 \in \Omega^+$, *where* Ω^+ *was defined in Eq.* 24, *then the optimization problem of Eq.* 25–30 *is guaranteed to be feasible for all times,* $x(t) \in \Omega^+$ *for all* $t \geq 0$ *and* $\limsup_{t\to\infty} \|x(t)\| \leq d$. *Furthermore, for* $x_0 \in \Pi^+ \backslash \Omega^+$ *where* Π^+ *was defined in Eq.* 23, *if the optimization problem of Eq.* 25–30 *is successively feasible for all times, then* $x(t) \in \Pi^+ \cup \Omega^+$ *for all* $t \geq 0$ *and* $\limsup_{t\to\infty} \|x(t)\| \leq d$.

Proof of Theorem 3: The proof of the theorem comprises of two parts. In part 1, we show the feasibility of the optimization problem for all $x \in \Omega^+$ and subsequent convergence to the desired neighborhood of the origin, while in part 2, for $x \notin \Omega^+$ we show convergence to the desired neighborhood of the origin upon assumption of feasibility of the optimization problem.

Part 1: From theorem 1 and the proof (see Ref. 39) it follows that given d, there exist positive real numbers δ' and Δ^*, such that if $\Delta \in (0, \Delta^*]$, then satisfaction of the constraints of Eqs. 27–28 ensures convergence to the desired neighborhood of the origin. In the proof, we show the existence of the positive real number ε^{**} (yielding Ω^+), which ensures initial and continued satisfaction of the constraints of Eqs. 27–28 for all $x_0 \in \Omega^+$. From the continuity of the functions $f(\cdot)$, $G(\cdot)$ $L_f V(\cdot)$ $L_G V(\cdot)$, the boundedness of u, and by restricting the state x_0 to the set Ω^+, it follows that given ε^* and Δ^* there exists a positive real number ε^{**}, such that if $L_f V(x_0) + L_G V(x_0) u_0 \leq -\varepsilon^{**}$ then $L_f V(x(\tau)) + L_G V(x(\tau)) u_0 \leq -\varepsilon^* \;\forall\, \tau \in (0, \Delta^*]$, where ε^*, Δ^* were defined in Theorem 1. This ensures initial feasibility of the constraints of Eq. 27

for all $x_0 \in \Omega^+$. Initial satisfaction of the constraints ensures that $V(x(t + \Delta)) \leq V(x(t))$, which in turn implies that $x(t+\Delta) \in \Omega^+$ for all $t \geq 0$, thereby yielding successive feasibility of the optimization problem. Successive feasibility of the optimization problem leads to convergence to the desired neighborhood of the origin.

Part 2: For all $x_0 \notin \Omega$, the assumption of initial and successive feasibility of the constraint of Eq. 29 ensures that $x(t + \tau) \in \Pi^+$ for all $x(t) \notin \Omega^+$, $\tau \in (0,\Delta^*]$. Also, the satisfaction of the constraint of Eq. 27 ensures that the value of the Lyapunov function continues to decrease, implying that the state trajectory eventually converges to the set Ω^+. Convergence to $\limsup_{t\to\infty} \|x(t)\| \leq d$ follow from part 1 previously. This concludes the proof of Theorem 3.

Remark 5: The meaning and implication of the constraints of Eqs. 27–29 is as follows: the constraint of Eq. 27 requires the control action to enforce a decay in the value of the Lyapunov-function value over the next time interval; because of the discrete nature (implement and hold) of the control action, such decay may not be achievable for all state values, and is only requested to drive the process to a desired neighborhood of the origin defined by $V(x) \leq \delta'$. Once the process reaches the desired neighborhood of the origin, the constraint of Eq. 28 prevents the state from escaping that neighborhood. For initial conditions within a level set of the Lyapunov function (Ω^+), successive decays in the Lyapunov function value is achievable and sufficient to drive the state to the desired neighborhood of the origin. For initial conditions outside the set Ω^+, the constraint of Eq. 29 asks for the control action to be computed such that for the process state at the next time instant, negative definiteness of $\dot{V}$ can be successively achieved. This ensures that out of all possible control actions that can achieve negative definiteness of $\dot{V}$, one is chosen that ensures that the state trajectory stays within Π^+ from where continued decay of the Lyapunov function value is possible. A continued decay in the Lyapunov function value leads to convergence to the desired neighborhood of the origin. Note also that in contrast to the result on linear system, guaranteed feasibility for all initial conditions in the null controllable region simply cannot be achieved, yet Eq. 29 represents a constraint that at least guides the control law to take some meaningful control action for initial conditions outside Ω^+. This constraint goes beyond (and does better than) simply requiring a decay in the value of the Lyapunov function and enables stabilization from a larger set of initial conditions (see the simulation example for a demonstration).

Remark 6: Note that if one were to use a bounded control design to achieve stability and characterize the stability region, one would also get the stability guarantees from an explicitly characterized set of initial conditions. However, the bounded controllers are designed to only achieve negative definiteness of $\dot{V}$. For initial conditions within Ω^+, decreasing value of the Lyapunov function ensures that the process stays within Ω^+ and is stabilized. For initial conditions outside Ω^+, while initially a decrease in the Lyapunov function value may be achieved, the process may drift out of the set of initial conditions from where $\dot{V} < 0$ is achievable. The bounded control designs do not have a mechanism to prevent this from happening, motivating the use of predictive control approach.

Remark 7: Note that the estimates of the stability region, and the enhancement with the proposed predictive controllers are influenced by the choice of the control Lyapunov function. Furthermore, referring to the choice of the control Lyapunov function (and this holds for other Lyapunov-based control laws as well), it is important to note that a general procedure for the construction of CLFs for nonlinear process systems of the form of Eq. 1 is currently not available. Yet, for several classes of nonlinear process systems that arise commonly in the modeling of engineering applications, it is possible to use suitable approximations,[43] or exploit system structure to construct CLFs. One approach that can be utilized in the construction of CLFs is to use the linearized system matrices to compute a quadratic Lyapunov function, and analyze the stability properties for the nonlinear system using the quadratic Lyapunov function. Possible conservatism in the stability region estimates can be mitigated by accounting for the stability region "size" in the choice of the Lyapunov function (parameterized by the entries in matrix P for a quadratic Lyapunov function of the form $x'Px$), for instance, by formulating an optimization problem to determine (if possible) a Lyapunov function whose derivative can be made negative definite over a desired neighborhood of the origin.

Remark 8: Note that in other predictive control approaches that pose a constraint for the state trajectory to go to some invariant set at the end of the horizon, or on determining appropriate cost functions that can serve as Lyapunov functions for the system, the horizon has to be "sufficiently" large (thereby increasing the complexity of computation) or one needs to assume feasibility of the optimization problem. A distinguishing feature of the proposed predictive controller is the formulation of stabilization constraints on the current control action which enables the characterization of the feasibility properties using Lyapunov-tools, without needing the horizon to be excessively large. While a rigorous analysis of the output-feedback problem (addressing the unavailability of some of the states as measurement), and robustness analysis remains outside the scope of this work, the proposed method can in principle be generalized to an output feedback setting, by incorporating robustness with respect to estimation errors in the state feedback controller and combining with an appropriately designed nonlinear observer. In this work, the robustness of the proposed predictive controller with respect to parametric uncertainty and disturbances are investigated via application to the styrene polymerization reactor.

Illustrative chemical process example. Consider a continuous stirred-tank reactor, where an irreversible, first-order exothermic reaction of the form $A \xrightarrow{k} B$ takes place. The mathematical model for the process takes the form

$$\begin{aligned}\dot{C}_A &= \frac{F}{V}(C_{A0} - C_A) - k_0 e^{\frac{-E}{RT_R}} C_A \\ \dot{T}_R &= \frac{F}{V}(T_{A0} - T_R) + \frac{(-\Delta H)}{\rho c_p} k_0 e^{\frac{-E}{RT_R}} C_A + \frac{Q}{\rho c_p V}\end{aligned} \tag{31}$$

where C_A denotes the concentration of the species A, T_R denotes the temperature of the reactor, Q is the heat added to the reactor, V is the volume of the reactor k_0, E, ΔH are the pre-exponential constant, the activation energy, and the enthalpy of the reaction, and c_p and ∂ are the heat capacity and

Table 2. Chemical Reactor Parameters and Steady–state Values

$V = 0.1$	m^3
$R = 8.314$	$KJKmol^{-1} \cdot K$
$C_{A0_s} = 1.0$	$Kmolm^{-3}$
$T_{A0_s} = 310.0$	K
$Q_s = 0.0$	$KJmin^{-1}$
$\Delta H = -4.78 \times 10^4$	$KJKmol^{-1}$
$k_0 = 72 \times 10^9$	min^{-1}
$E = 8.314 \times 10^4$	$KJKmol^{-1}$
$c_p = 0.239$	$KJKg \cdot K^{-1}$
$\rho = 1000.0$	Kgm^{-3}
$F = 100 \times 10^{-3}$	$m^3 min^{-1}$
$T_{Rs} = 395.33$	K
$C_{As} = 0.57$	$Kmolm^{-3}$

fluid density in the reactor. The values of all process parameters can be found in Table 2. The control objective is to stabilize the reactor at the unstable equilibrium point $(C_A^s, T_R^s) = (0.57 \text{ Kmol/m}^3, 395.3 \text{ K})$ using the rate of heat input, Q, and change in inlet concentration of species A, $\Delta C_A = C_{A0} - C_{A0_s}$ as manipulated inputs with constraints: $|Q| \leq 32$ KJ/s and $|\Delta C_{A0}| \leq 1$ Kmol/m^3.

We first construct a Lyapunov-based predictive controller using a $V(x) = x'Px$, where $x = (C_A - C_A^s, T_R - T_R^s)$, $P = \begin{pmatrix} 0.983 & 0.025 \\ 0.025 & 0.001 \end{pmatrix}$, where the matrix P is computed by solving the Riccati inequality with the linearized system matrices. The parameters in the objective function of Eq. 30 are chosen as $Q = qI$, with $q = 0.1$, and $R = \begin{pmatrix} 10.0 & 0.0 \\ 0.0 & 10000.0 \end{pmatrix}$. The set Π and the stability region estimate under the Lyapunov-based controller Ω are computed and shown in Figure 3. The constrained nonlinear optimization problem is solved using the MATLAB subroutine FMINCON, and the set of ODEs is integrated using the MATLAB solver ODE15s.

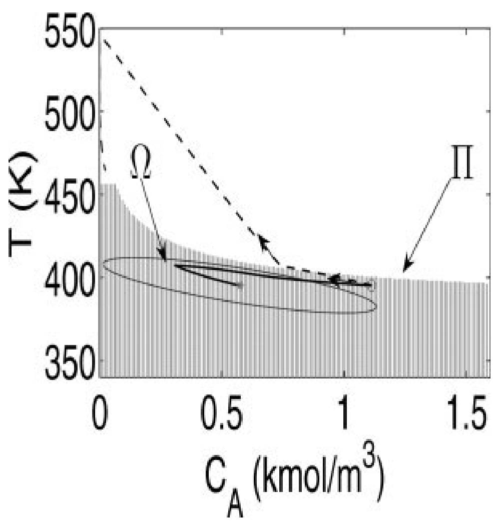

Figure 3. Evolution of the state trajectory for the chemical reactor example under the predictive controller of Eqs. 3–7 (dashed line) with a stability region Ω, and under the proposed predictive controller (solid line) enabling stabilization from initial conditions outside Ω.

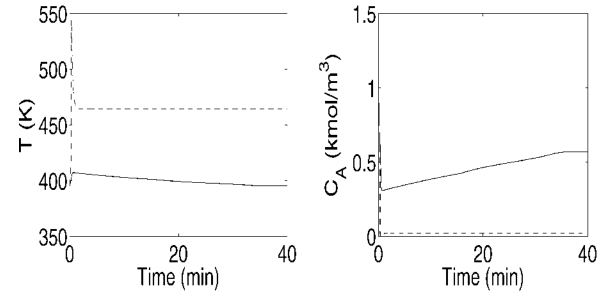

Figure 4. The state profiles for the chemical reactor example under the predictive controller of Eqs. 3–7 (dashed lines), and under the proposed predictive controller (solid lines) from an initial condition outside Ω.

To illustrate the enhancement in the set of initial conditions from where closed–loop stability can be achieved using the proposed controller, we pick an initial condition $C_A(0)$, $T_R(0) = 1.113$ kmol/m^3, 395.3 K outside Ω^+, but inside Π^+. We first implement the Lyapunov-based predictive controller of Theorem 1 that only requires the value of the Lyapunov function to decrease. Since the initial condition is within the set Π^+, there exists a control action that can enforce negative definiteness of the Lyapunov function, and the controller proceeds to implement such control action. However, enforcing negative definiteness of $\dot{V}$ (i.e., driving the trajectory to successively lower level curves of the Lyapunov function), is not sufficient to ensure that the trajectory remains within the set Π^+. At $t = 0.12$ min, the state trajectory escapes out of Π^+, and it is no longer possible to find a control action that enforces negative definiteness of $\dot{V}$. If the stability constraints are removed to allow feasibility of the optimization problem, the value of the Lyapunov function continues to increase (see dashed lines in Figures 4 and 5 for the corresponding state and input profiles), and closed–loop stability is not achieved. In contrast, if the proposed predictive controller is implemented, it not only enforces negative definiteness of $\dot{V}$, but also ensures that the state trajectory does not escape Π^+. In other words, out of possible state trajectories along decreasing values of the level curves of $V(x)$, those are chosen (if they exist) that keep the state profile in Π^+. Closed–

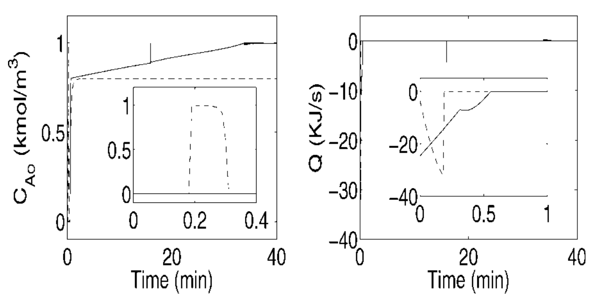

Figure 5. The input profiles for the chemical reactor example under the predictive controller of Eqs. 3–7 (dashed lines), and under the proposed predictive controller (solid lines) from an initial condition outside Ω.

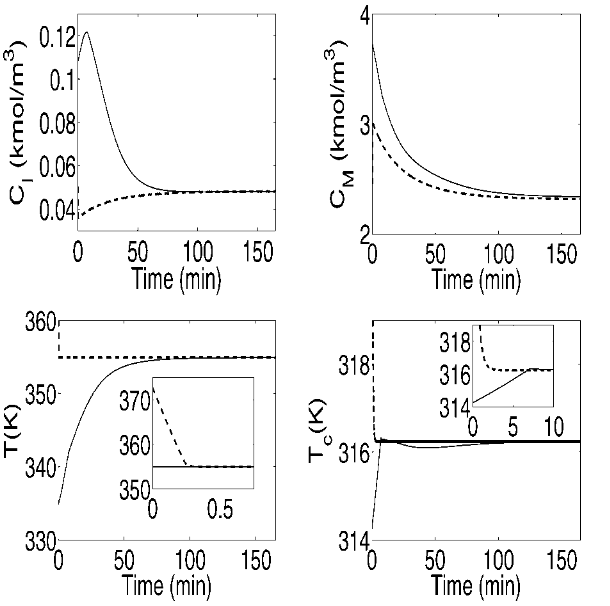

Figure 6. Evolution of the state profiles for the styrene polymerization process under the proposed predictive controller in the absence (solid lines) and presence of disturbances (dashed lines).

loop stability is, thereby, achieved, demonstrating an expansion on the set of initial conditions from where closed–loop stability can be achieved by better utilizing the constraint enforcing capabilities of the predictive control approach.

Application to the Styrene Polymerization Process

Having illustrated the enhancement in the stability region via a chemical reactor example, we implement the predictive controller on the styrene polymerization process. To this end, first a quadratic Lyapunov function of the form $V(x) = x^T Px$ with $P = \begin{bmatrix} 52570 & 2457 & 261.4 & 6.942 \\ 2457 & 181.9 & 13.40 & 2.561 \\ 261.4 & 1.340 & 1.708 & 0.2300 \\ 6.942 & 2.561 & 0.2300 & 0.9668 \end{bmatrix}$ is chosen in the predictive controller design and the set of initial conditions from where $\dot{V} < 0$ is achievable (the set Π^+), and the invariant set Ω^+ within Π^+ (defined by $V(x) \leq 105$) is computed. In the application, the "discretized" version of the stability constraint are implemented, i.e., $V(x(t + \Delta)) < V(x(t))$ is implemented instead of $\dot{V}(x(t + \tau)) < -\varepsilon^*$, and $x(t + \Delta) \in \Pi^+$ is implemented instead of $x(t + \tau) \in \Pi^+$. The weighting matrices in the predictive controller were chosen as $Q = qI_x$ with $q = 0.1$ and $R = rI_u$ with $r = 1$, where I_x and I_u are the identity matrices of appropriate dimensions.

We first demonstrate the implementation of the predictive control algorithm for an initial condition ($C_I(0) = 0.11$ $kmolm^{-3}$, $C_M(0) = 3.73$ $kmolm^{-3}$, $T(0) = 334.92$ K, $T_c(0) = 314.24$ K) outside the set Ω^+. As can be seen from the solid lines in Figures 6 and 7 (which show the evolution of the state and input profiles), even though the initial condition is significantly outside the stability region estimate, the predictive controller is able to stabilize the closed–loop system. We next investigate the robustness of the predictive controller with respect to parametric uncertainty and disturbances from an initial condition $C_I(0) = 0.05$ $kmolm^{-3}$, $C_M(0) = 2.45$ $kmolm^{-3}$, $T(0) = 372.66$ K, $T_c(0) = 332.05$ K. Specifically, we consider errors in the values of the parameters A_p, hA and V_c of magnitude 1%, 2% and 10%, respectively, as well as sinusoidal disturbances in the initiator flow rate F_i, and the coolant inlet temperature T_{cf} of magnitude 10% around the nominal values. These parametric errors and disturbances result in a change in the value of the nominal steady state. The predictive controller is, however, able to offset the effect of the parametric errors and disturbances demonstrating robust stabilization of the closed–loop system (see dashed lines in Figures 6 and 7).

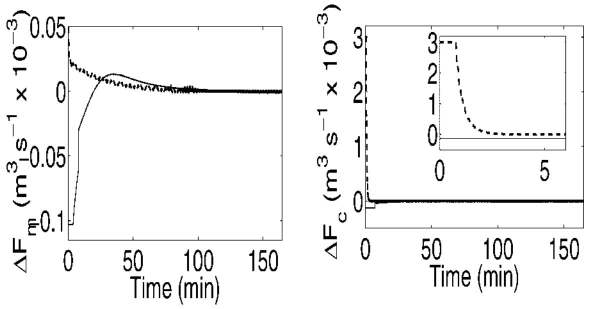

Figure 7. The input profiles showing the deviations from the nominal monomer and coolant flow rates for the styrene polymerization process under the proposed predictive controller in the absence (solid lines) and presence of disturbances (dashed lines).

Conclusions

This work considered the problem of predictive control of nonlinear process systems subject to input constraints. A predictive controller for linear systems was first designed that achieves stability for every initial condition in the null controllable region without resorting to infinite horizons. For nonlinear process systems, predictive controllers were designed that expand on the set of initial conditions from where closed–loop stability is achievable. The proposed method was illustrated using a chemical reactor example, and the robustness with respect to parametric uncertainty and disturbances demonstrated via application to a styrene polymerization process.

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